Human Capital and Popular Investment Advice*

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Abstract

Popular investment advice recommends that the stock/bond and stock/wealth ratios should rise with investor risk tolerance and investment horizon respectively, prescriptions that are difficult to reconcile with the simple mean-variance model. We show that extending the mean-variance model to include human capital, without any other modifications, can simultaneously justify both recommendations, so long as the correlation between labour income and stock returns falls within a range determined by market and investor-specific parameters. Aggregate labour income data from 11 countries generally satisfy this requirement, as do plausible individual income processes. We also consider the implications of human capital for the optimal bond/wealth ratio over the investment horizon, and examine the sensitivity of the stock/bond mix to the volatility of labour income.

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I Introduction

One of the cornerstones of modern portfolio theory is the two-fund separation theorem. This seminal result, originally due to Tobin (1958), states that the composition of the optimal portfolio of risky assets depends solely on the stochastic structure of market returns and is independent of investor-specific characteristics such as risk tolerance. However, Canner et al. (1997) document investment advice recommending that less risk-tolerant investors hold a higher ratio of bonds to stocks, a phenomenon they refer to as the asset allocation puzzle. As they point out, this is perplexing not only because the advice differs from that implied by theory, but also because it is more complicated than theory.[1]

For many investors, human capital is a significant part of their overall portfolio, but it plays no role in the one-period mean-variance model that gives rise to the two-fund separation theorem.[2] Extending the model to incorporate human capital can potentially resolve the asset allocation puzzle. To see this, suppose that returns to human capital are perfectly correlated with those on stocks. Then human capital and stocks are perfect substitutes, so the separation theorem implies that the ratio

$$\frac{\text{BONDS}}{\text{HUMAN CAPITAL} + \text{STOCKS}}$$

is independent of investor risk tolerance. An increase in risk tolerance increases both the numerator and denominator of this ratio in the same proportion. But at any point in time, the quantity of human capital is non-tradable and thus fixed, so the quantity of stocks must rise proportionately more than the quantity of bonds. That is, the ratio of stocks to bonds is greater for more risk-tolerant investors, just as popular advice recommends.

However, Canner et al. (1997) reject this explanation on two grounds. First, human capital and stocks are unlikely to be perfect substitutes for many investors. Second, if human capital returns are strongly correlated with stock returns, then it becomes difficult to reconcile theory with another popular piece of investment advice: that young investors with long investment horizons should hold more of their wealth in stocks than older investors.[3] In general, young investors have more human capital than their older counterparts, so a high positive correlation

[1]This advice also seems to be followed in practice. Degeorge et al. (2004) examine the participation decisions of France Telecom employees in that firm's 1997 privatization share offering and find that civil servant workers invest proportionately more in the most "bond-like" vehicle than do their (plausibly less risk averse) private sector counterparts. Other proxies for risk aversion yield similar patterns.

[2]Nevertheless, the potential importance of human capital for financial decisions has long been recognized. For example, Mayers (1972) shows that the presence of non-marketable assets such as human capital introduce an extra term into the CAPM risk premium. More recently, Bodie et al. (1992), Heaton and Lucas (1997), and Viceira (2001), among others, examine the role of labour and business income for various portfolio decisions.

[3]A simple rule of thumb recommends that the portfolio percentage devoted to stocks should equal 100 minus

between stocks and human capital implies that young investors optimally allocate less of their wealth to stocks, thereby contradicting the standard advice. By contrast, if stocks are a good hedge for human capital, then younger investors should indeed hold a higher proportion of their wealth in stocks, just as popular advice dictates. But then the advice that more risk-tolerant investors hold a higher ratio of stocks to bonds cannot be explained.

Thus, human capital considerations seem unable to resolve the asset allocation puzzle without simultaneously creating another equally perplexing puzzle. On the one hand, the strong correlation between human capital returns and stock returns required to justify the asset allocation advice makes the investment horizon advice more puzzling. On the other hand, the weak correlation between human capital returns and stock returns required to justify the investment horizon advice exacerbates the asset allocation puzzle. More succinctly, it seems impossible for the human capital of any investor to simultaneously justify both pieces of advice.[4]

One explanation for this conundrum is that investment advisors are simply wrong.[5] However, such a pessimistic conclusion warrants further scrutiny. Specifically, we ask two questions. First, despite the misgivings outlined above, is it theoretically possible for human capital considerations to reconcile the simple mean-variance model with popular advice on asset allocation and investment horizon? Second, if such a theoretical explanation does exist, is it empirically plausible? That is, given the ubiquitous nature of these investment recommendations, do the conditions required for human capital to offer an explanation in theory seem likely to also exist in practice?

For the first question, we use a simple extension of the Campbell and Viceira (2002) log-linear version of the mean-variance model, and show that human capital factors *can* justify popular advice about both asset allocation and investment horizon decisions so long as the correlation between stock and human capital returns falls within some range defined by market and investor-specific parameters. To address the second question, we use historical data on aggregate asset returns and labour income from 11 countries to estimate these parameters and find that, at least for the various data sets we employ, the stock-human capital correlation generally falls within the allowed range. We offer some tentative evidence suggesting that this is also likely to be true

the investor's age; see also the Vanguard Group advice quoted in Ameriks and Zeldes (2001). This process is sometimes known as 'time diversification'; see Kritzman (1994) and Jagannathan and Kocherlakota (1996) for particularly lucid discussions.

[4]Of course, one could argue that the investment horizon advice is justified by other considerations. For example, long-term stock returns could be mean-reverting, so that stocks are less risky over a long investment horizon. However, in 75 years of data from 30 countries, Jorion (2003) finds little evidence of this. Moreover, Jagannathan and Kocherlakota (1996) argue that human capital considerations represent the only convincing explanation for the view that the stock/wealth ratio should rise with investment horizon.

[5]See Anonymous (1997) for an example of this interpretation.

for most individual labour income processes.

Previous research has identified other possible solutions for the asset allocation puzzle. Elton and Gruber (2000) argue that theory and popular advice can be reconciled by introducing various constraints into the mean-variance model, while Shalit and Yitzhaki (2003) suggest that the advice is not necessarily inefficient for alternative investor preferences. In addition, Brennan and Xia (2000) and Campbell and Viceira (2001) show that time-varying expected returns can justify the advice for an infinitely-lived investor. None of these, however, considers the relevance of their analysis for investment horizon considerations. Other authors, such as Bodie et al. (1992), Jagannathan and Kocherlakota (1996), and Viceira (2001), show that human capital considerations can justify the popular investment horizon advice, but do not discuss the asset allocation puzzle. None of this work, therefore, considers whether or not recognition of non-tradable human capital can simultaneously justify *both* pieces of investment advice.

An interesting exception to this is Gomes and Michaelides (2002), who calibrate a multi-period model with non-mean-variance preferences and a fixed market entry cost and find optimal behaviour that is broadly consistent with both pieces of investment advice. Our work differs from theirs in two ways. First, their primary focus is on other matters, so they do not explore the source of this consistency in any detail. Second, and more importantly, their model is much more complex than ours. The primary contribution of our work is to show that inclusion of human capital in the simple mean-variance model, *without any further modifications*, can potentially resolve the investment advice puzzles associated with that model. Although the dynamic effects associated with multi-period decision-making, for example, are undoubtedly important for explaining other types of investment behaviour and practice, our analysis indicates that they are not essential for explaining the puzzle documented in this paper.

In the next two sections, we outline our model and apply it to the investment advice puzzle. Section IV contains the results of our calibration exercise. In Section V, we go beyond existing investment advice and apply our model to some additional questions. If it is indeed optimal to reduce stock holdings as the investment horizon shortens, what happens to optimal bond holdings? And does the optimal ratio of stocks to bonds depend on the volatility of labour income? Section VI contains some concluding remarks.

II Optimal asset allocation in the presence of non-tradable human capital

An investor has some initial endowment of financial wealth $\overline{W} > 0$ which is used to construct a portfolio that generates the random rate of return R_p. One period later, the investor consumes the portfolio's liquidation value $\overline{W}(1+R_p)$ and labour income L earned over the period.

At the beginning of the period, the investor chooses the portfolio that maximizes the expected utility of terminal wealth $W = \overline{W}(1+R_p)+L$. This decision is determined by the power utility function

$$U(W) = \frac{W^{1-\gamma}}{1-\gamma}, \tag{1}$$

where $\gamma > 0$ is the coefficient of relative risk aversion.

In Campbell and Viceira (2002), the investor's portfolio decision consists of choosing the optimal combination of two assets, one of which is riskless while the other is risky. To address the asset allocation puzzle of Canner et al. (1997), we require two risky assets, so we extend the Campbell and Viceira model to a three-asset setting. Asset f, which we call cash, is riskless and offers the rate of return R_f over the period. Asset s, which we call stocks, is risky with a random rate of return R_s. Asset b, which we call bonds, is also risky and has a random rate of return R_b. We assume that $1+R_s$, $1+R_b$, and labour income L are lognormal random variables.

The portfolio shares allocated to assets s and b are α_s and α_b respectively. Thus, the investor chooses these portfolio shares to maximize the expected value of (1) subject to the budget constraint

$$W = \overline{W}(1+R_p) + L, \tag{2}$$

where $R_p = R_f + \alpha_s(R_s - R_f) + \alpha_b(R_b - R_f)$. This problem is nonlinear, so we apply some linear approximations that effectively reduce it to a mean-variance setting. The details of this procedure are straightforward, but tedious, so we relegate them to an appendix. There we show that the optimal asset allocations are

$$\alpha_s = \frac{1}{\Delta}\left(\frac{1}{\rho\gamma}\left(\sigma_b^2\mu_s - \sigma_{sb}\mu_b\right) + \left(1-\frac{1}{\rho}\right)\left(\sigma_b^2\sigma_{ls} - \sigma_{sb}\sigma_{lb}\right)\right), \tag{3}$$

$$\alpha_b = \frac{1}{\Delta}\left(\frac{1}{\rho\gamma}\left(\sigma_s^2\mu_b - \sigma_{sb}\mu_s\right) + \left(1-\frac{1}{\rho}\right)\left(\sigma_s^2\sigma_{lb} - \sigma_{sb}\sigma_{ls}\right)\right), \tag{4}$$

where, for $i = s, b$, $l = \log L$, $r_i = \log(1+R_i)$, $\sigma_i^2 = \mathrm{Var}[r_i]$, $\sigma_{sb} = \mathrm{Cov}[r_s, r_b]$, $\sigma_{li} = \mathrm{Cov}[l, r_i]$, $\Delta = \sigma_s^2\sigma_b^2 - \sigma_{sb}^2$ is the determinant of the variance-covariance matrix, $\mu_i = E[r_i] - r_f + \sigma_i^2/2$ is the logarithmic risk premium for asset i, and $\rho \in (0,1)$ is a linearization parameter that is defined in detail in the appendix.

Note that without labour income, the second terms in the large brackets in (3) and (4) both equal zero, so the ratio α_s/α_b is independent of investor risk attitudes γ, i.e., the two-fund separation theorem applies. With labour income, however, this independence disappears. In the next section, we determine whether this can change the model's implications in a way that is consistent with popular investment advice.

In doing so, the term

$$\phi_p \equiv \alpha_b \frac{\partial E[r_p]}{\partial \alpha_b} + \alpha_s \frac{\partial E[r_p]}{\partial \alpha_s}$$

often appears, where $E[r_p] = E[\log(1 + R_p)]$ is the expected log return on the chosen portfolio. ϕ_p thus represents the effect on the expected log portfolio return of shifting investment funds from the riskless asset to the risky asset portfolio. Such a shift obviously increases the expected simple portfolio return $E[R_p]$ so long as the individual asset risk-premia are positive, but the log counterpart $E[r_p]$ is a quadratic function of the portfolio weights, so the sign of ϕ_p is ambiguous. In the appendix, we show that

$$\phi_p = \mathrm{Var}[r_p](\gamma(\rho + (1 - \rho)\beta) - 1), \tag{5}$$

where $\beta = \mathrm{Cov}[r_p, l]/\mathrm{Var}[r_p]$. Intuitively, when labour income risk is idiosyncratic ($\beta = 0$), investors with $\gamma\rho = 1$ hold the portfolio with the maximum expected log portfolio return, so less conservative investors ($\gamma\rho < 1$) who allocate more of their portfolio to risky assets must have a lower $E[r_p]$, i.e., ϕ_p is negative for such investors. When labour income risk is systematic, investors with $\gamma\rho = 1$ allocate more or less to risky assets, depending on whether risky assets are a good or poor hedge against labour income shocks. In general, ϕ_p is positive so long as investor risk aversion is not too low and labour income is not too negatively correlated with risky asset returns.

In what follows, we focus on the case where ϕ_p is positive, for the following reasons. First, most existing evidence suggests that most investors have high γ and a labour income stream that has positive, or at least not strongly negative, correlation with risky asset returns. Second, ϕ_p is always positive in the data we use below. Third, any reversal of our results would require ϕ_p to be strongly, and implausibly, negative, so focusing on cases where it is positive sacrifices little generality while considerably simplifying the accompanying discussion.

III Risk aversion, investment horizon, and optimal asset choice

We first determine the effect of risk aversion γ on the ratio $\alpha \equiv \alpha_s/\alpha_b$. According to the two-fund separation theorem, α and γ are independent, but popular advice, as documented in Canner et al. (1997), recommends that less risk-tolerant investors hold a lower ratio of stocks to bonds. That is, $\partial\alpha/\partial\gamma$ should be negative.

In our model, $d\alpha/d\gamma$ has the same sign as (see the appendix)

$$(1 + \phi_p)(\mu_s \sigma_{lb} - \mu_b \sigma_{ls}). \tag{6}$$

Let $c_{li} = \sigma_{li}/\sigma_l\sigma_i$ denote the linear correlation coefficient for human capital and asset i returns.

Then so long as $\phi_p > -1$, (6) is negative if and only if

$$c_{ls} > \underline{H}, \tag{7}$$

where $\underline{H} = c_{lb}(\mu_s/\sigma_s)/(\mu_b/\sigma_b)$.[6] Thus, more risk-tolerant investors should indeed hold a higher proportion of stocks, so long as their labour income is sufficiently strongly correlated with stock returns, the required extent of which is determined by the relative size of the stock and bond Sharpe ratios. This condition reflects the balancing of the two determinants of asset demand: the ability to hedge human capital returns and the risk-return trade-off as measured by the Sharpe ratio. If (7) is satisfied, the hedging capabilities of the bond are sufficient to offset the risk-return properties of the stock, so investors who wish to reduce their risk exposure hold less of both stocks and bonds, but reduce stock holdings by more since they must continue to hold their non-tradable human capital.[7]

This result is a simple extension of the Canner et al. (1997) argument that human capital considerations can justify popular asset allocation advice if stocks and human capital are perfect substitutes. It shows that human capital need only be relatively more "stock-like" than "bond-like", thereby negating Canner et al.'s concern that perfect substitutability is unlikely to be the case for most investors. What remains unresolved is whether this weaker condition can also overcome Canner et al.'s other, more important, objection: that relatively "stock-like" human capital is inconsistent with popular advice on the relationship between investment horizon and optimal stock holdings. This advice is neatly summarized by Malkiel (1996):

> "...the longer the time period over which you can hold on to your investments, the greater should be the share of common stocks in your portfolio."

However, if human capital is strongly correlated with the stock market, then the holding of "stock-like" assets automatically increases with the investment horizon, thereby implying that the share of traded stocks should be smaller the longer the time period over which the portfolio can be held. If human capital considerations are to be a plausible explanation for the asset allocation puzzle, then the required condition (7) should not rule out the recommended relationship between stock holdings and investment horizon, i.e., the required high correlation between stock market returns and labour income should not be so high as to imply that investors with a long time horizon should hold a lower share of common stocks in their portfolios.

[6] To avoid unnecessary complications associated with negative numbers, we anticipate our subsequent empirical findings and assume $\mu_b > 0$.

[7] If $\phi_p < -1$, then the risky asset portfolio is a particularly good hedge against labour income risk (see (5)), so investors who wish to reduce their risk exposure hold *more* risky assets, and, in particular, hold relatively more of the risky asset most like human capital since their direct holdings of the latter are fixed.

To address this issue, we use the parameter $z \equiv E[\log(L/\overline{W})]$ as a proxy for the length of investment horizon. This can be justified by noting that young investors, with long investment horizons and long working lives, have high expected future labour income but low financial wealth, so they have higher z than do older investors with shorter investment horizons.[8]

In our model, $d\alpha_s/dz$ has the same sign as (see the appendix)

$$\sigma_b^2(\mu_s - \gamma\sigma_{ls}) - \sigma_{sb}(\mu_b - \gamma\sigma_{lb}). \tag{8}$$

To justify popular advice, this expression must be positive. This occurs if and only if

$$c_{ls} < \overline{H}, \tag{9}$$

where $\overline{H} = (1/\gamma\sigma_l\sigma_s)(\mu_s - (c_{sb}\sigma_s/\sigma_b)\mu_b) + c_{sb}c_{lb}$. Thus, long-horizon investors should indeed allocate a greater proportion of their wealth to stocks so long as the correlation between stock returns and labour income is not too high.[9] The condition appearing in (9) gives concrete expression to what is meant by "not too high". If (9) is satisfied, then stocks are a good hedge for non-tradable human capital, so a young investor with a long investment horizon puts more into stocks than an older investor with a shorter horizon, just as popular advice recommends.

Of course, what we are primarily interested in is whether the joint distribution of labour income and stock and bond returns can justify popular advice in relation to both asset allocation and investment horizon; that is, whether (7) and (9) can hold simultaneously. This occurs if and only if

$$\underline{H} < c_{ls} < \overline{H}. \tag{10}$$

Thus, so long as the correlation between stock returns and labour income lies between two bounds, investors should allocate less of their wealth to stocks as their investment horizon shortens and they should adjust their bond/stock ratios downwards in response to any increase in tolerance for risk.

Various parameter combinations would automatically disqualify this requirement. For example, if the stochastic structure of asset returns and labour income were such that $\underline{H} \geq \overline{H}$, or $\overline{H} \leq -1$, or $\underline{H} \geq 1$, then (10) cannot hold. However, inspection of $\underline{H}$ and $\overline{H}$ reveals that there are combinations of parameters for which (10) is satisfied, so it is theoretically possible for human capital considerations to reconcile the simple mean-variance model with popular advice on asset allocation in a way that does not conflict with other popular advice on the relationship between stock holdings and the investment horizon. What remains unclear is whether such a possibility is empirically plausible; that is, whether *actual* parameter values satisfy (10).

[8]Young investors also have greater future liabilities (consumption) than their older counterparts, but this important difference cannot be captured in a static model.

[9]See Bodie et al. (1992) and Jagannathan and Kocherlatoka (1996) for a similar conclusion.

Table 1: U.S. estimates of c_{ls}, $\underline{H}$, and $\overline{H}$

Popular investment advice on (i) stock/bond allocation and risk tolerance and (ii) stock/wealth allocation and investment horizon can both be justified if and only if $\underline{H} < c_{ls} < \overline{H}$, where c_{ls} is the linear correlation between labour income and stock returns, and $\underline{H}$ and $\overline{H}$ are constants that depend on market and investor-specific parameters. For the U.S., for each year in the period 1930–1999, market return data are obtained from Ibbotson Associates and labour income data from the Department of Commerce's Bureau of Economic Analysis. γ is the coefficient of relative risk aversion. Based on a Wald test, * indicates that $c_{ls} - \underline{H}$ (or $\overline{H} - c_{ls}$) is positive at the 10% significance level, ** that it is positive at the 5% level, and *** that it is positive at the 1% level.

Bond type	Sample		c_{ls}	$\underline{H}$		$\overline{H}$		γ for which
	Begin	End			$\gamma = 2$	$\gamma = 5$	$\gamma = 10$	$c_{ls} < \overline{H}$
Long-term corporate bonds	1930	1999	0.11	−0.02	4.34^{***}	1.74^{***}	0.87^{***}	$0 < \gamma < 75.4$
Long-term government bonds	1930	1999	0.11	−0.12	4.66^{***}	1.86^{***}	0.92^{***}	$0 < \gamma < 75.1$
Intermediate government bonds	1930	1999	0.11	0.03	5.05^{***}	2.02^{***}	1.01^{***}	$0 < \gamma < 91.5$

IV Empirical estimates of c_{ls}, $\underline{H}$, and $\overline{H}$

To determine whether our model's justification of popular advice is plausible, we use historical data to estimate c_{ls}, $\underline{H}$, and $\overline{H}$. If these estimates satisfy (10), then this is consistent with the view that popular investment advice implicitly incorporates human capital considerations; failure to satisfy (10) suggests that popular investment advice cannot be justified by human capital considerations, at least not in the way envisaged by our model.

To obtain estimates of the terms in (10), we employ time series of asset returns and aggregate labour income data. Initially, we use per-capita income data from the U.S. Department of Commerce's Bureau of Economic Analysis (BEA) for the period 1930–99, together with annual U.S. market returns data from Ibbotson Associates.[10] Ibbotsons report real returns for long-term government bonds, intermediate-term government bonds, and corporate bonds, so we calculate $\underline{H}$ and $\overline{H}$ for each bond type; real stock returns are calculated from the large company index.

With these data, we calculate estimates of the various means, standard deviations, and correlations that appear in equations (3) and (4). We then substitute these estimates into the terms appearing in (10), a process that yields the results in Table 1. Focusing first on the difference between $\underline{H}$ and c_{ls}, the labour-stock correlation is 0.11, but the estimates of $\underline{H}$ range from −0.12 to 0.03. Thus, regardless of the class of bond, the first inequality in (10) is

[10]The BEA data are available from http://www.bea.doc.gov/. Ideally, we would use actual human capital data, but we are unable to locate reliable sources of this variable for most of the countries we subsequently examine. Since labour income is equal to human capital in a one-period world, we use the former as a proxy for the latter.

satisfied. Turning to the second inequality, we report estimates of $\overline{H}$ for low, medium, and high values of γ ($\gamma = 2, 5, 10$ respectively).[11] Most of these estimates are greater than unity, thereby automatically exceeding c_{ls}; even the smallest estimate of $\overline{H}$ is almost eight times as large as the labour-stock correlation. In the final column of Table 1, we express this result in a different way by reporting the range of γ values for which $\overline{H} - c_{ls}$ is positive; for this not to occur, γ must attain at least the implausibly-high value of 75.[12]

While these results are consistent with our model, some important caveats apply. To begin with, the terms in (10) depend in part on means and correlations, parameters that are notoriously difficult to estimate with any precision. Thus, the point estimates appearing in Table 1 are likely to be subject to considerable error.

To address this issue, we use a Wald test of both inequalities in (10).[13] For the first inequality, we define

$$h_1 = \mu_b \sigma_{ls} - \mu_s \sigma_{lb}$$

and test the null hypothesis that $h_1 \leq 0$ against the alternative that $h_1 > 0$. For the second inequality, we define

$$h_2 = \sigma_b^2 \mu_s - \sigma_{sb} \mu_b - \gamma(\sigma_b^2 \sigma_{ls} - \sigma_{sb} \sigma_{lb})$$

and test the null hypothesis that $h_2 \leq 0$ against the alternative that $h_2 > 0$. In either case, rejecting the null supports the corresponding inequality in (10).

The results of this test procedure also appear in Table 1 and provide both good and bad news for our story. The good news is that h_2 is positive at the 1% significance level in all cases, even at the highest level of risk aversion. The bad news is that h_1 is insignificantly different from zero in all cases. This difference reflects the fact that the estimated standard errors for $\overline{H}$ and c_{ls} are quite small, whereas that for $\underline{H}$ is large.[14] Thus, our U.S. data strongly support the notion that human capital considerations can explain the investment horizon advice, but are rather more reticent about the asset allocation advice.

Another possible reservation about our results is that they apply to a single country only, yet the investment advice seems to be an international phenomenon. To examine this issue, we

[11] Because $\sigma_b^2 \mu_s - \sigma_{sb} \mu_b > 0$ in our data, $\overline{H}$ is monotonically decreasing in γ.

[12] We also calculate c_{ls}, $\underline{H}$, and $\overline{H}$ assuming that asset returns are lagged one year, reflecting possible lags in labour income (see Campbell et al., 2001). Although this results in different estimates of c_{ls} and $\underline{H}$ individually, it has virtually no effect on the difference between them. Similarly, the difference between $\overline{H}$ and c_{ls} remains large and positive.

[13] See, for example, Greene (1993, pp. 131–133).

[14] Indeed, in order for the point estimates in Table 1 to be able to reject the null that $h_1 \leq 0$ at the 5% significance level, we would need approximately 350 years of data in the case of long-term government bonds, and even more for the other two bond categories.

estimate the terms in (10) with data from other countries. For asset returns, we use the data series generated by Dimson et al. (2002); these contain annual real returns on equities, bonds and bills from 1901 to 2002 for 15 countries (not including the U.S.).[15] For labour income, we use International Financial Statistics data published by the IMF, deflating these nominal series by their corresponding CPI values. The income series are of shorter duration than the Dimson et al. returns series and, moreover, are not available for all 15 countries. In all, we are able to calibrate our model with data from 10 additional countries over post-WWII periods of varying length. For completeness, we also include the U.S. to check that our earlier results are not sensitive to the source of labour income data.

The results from using these data appear in Table 2. Several features are apparent. First, the post-WWII results for the U.S. are very similar to those for the longer time period reported in Table 1. Second, for eight of the other 10 countries, the bounds specified by (10) are satisfied, consistent with popular investment advice being motivated by human capital considerations. Moreover, in most of these eight countries, c_{ls} differs from its two bounds by fairly large margins. In the two countries (Canada and Italy) where the first bound is violated, the difference is small. Third, c_{ls} is less than $\overline{H}$ at conventional significance levels in all countries for all reported values of γ. Moreover, $c_{ls} \geq \overline{H}$ only for implausibly-high risk aversion. Fourth, c_{ls} is greater than $\underline{H}$ at conventional significance levels in three countries (France, Japan, and the U.K.).

Overall, our international data paint much the same picture as U.S. data: they provide strong support for the link envisaged by our model between human capital and investment horizon advice, but statistically weaker support for the human capital link with asset allocation advice.

A third area of concern is that our labour income parameter values are based on aggregate per-capita income data and thus may differ greatly from the parameter values of actual individual investors. In particular, macroeconomic stabilization policies and the levelling inherent in aggregation suggest that individual labour income processes are likely to be much less smooth than the aggregate per-capita process. Moreover, individual labour income characteristics exhibit considerable cross-sectional dispersion and the ubiquitous and unconditional nature of the investment advice suggests that it should apply to all investors, not just those whose labour income process is similar to that implied by aggregate data. These issues imply a need to consider the possible effect on our results of labour income that is both more volatile and more correlated with stock returns.

Focusing first on correlations, note that higher c_{ls} alone makes it *more* likely that the critical value $\underline{H}$ is exceeded. Thus, greater individual correlations make it easier to justify the asset

[15] These data are available from Ibbotson Associates.

Table 2: International estimates of c_{ls}, $\underline{H}$, and $\overline{H}$

This table repeats the calculations of Table 1, but for alternative data sets. Market returns are calculated from the real bond, equity, and bill indices in Dimson et al. (2002). Labour income for each country is generated using IMF labour income data, deflated by the respective CPI series. As in Table 1, *, **, and *** indicate that the corresponding bound on c_{ls} in (10) is statistically significant (using a Wald test) at the 10%, 5%, and 1% significance levels respectively.

Country	Sample		c_{ls}	$\underline{H}$	$\overline{H}$			γ for which
	Begin	End			$\gamma = 2$	$\gamma = 5$	$\gamma = 10$	$c_{ls} < \overline{H}$
Australia	1963	2002	-0.29	-0.68	7.82^{**}	3.11^{**}	1.53^{**}	$\gamma > 0$
Canada	1949	2002	-0.00	0.09	10.15^{**}	4.06^{**}	2.03^{**}	$\gamma > 0$
France	1950	2002	0.14	-0.07^{*}	4.85^{*}	1.92^{*}	0.94^{*}	$0 < \gamma < 57.3$
Ireland	1949	2002	0.38	0.08	7.81^{***}	3.13^{***}	1.57^{**}	$0 < \gamma < 42.0$
Italy	1960	2002	-0.25	-0.21	3.16^{*}	1.24^{*}	0.60^{*}	$\gamma > 0$
Japan	1949	2002	0.34	-0.18^{*}	4.46^{***}	1.78^{***}	0.89^{**}	$0 < \gamma < 26.0$
Netherlands	1950	2002	-0.02	-0.12	9.38^{***}	3.75^{***}	1.88^{***}	$\gamma > 0$
Spain	1961	2002	-0.13	-0.52	3.77^{**}	1.43^{**}	0.65^{**}	$0 < \gamma < 826.8$
Sweden	1961	2002	-0.03	-0.08	6.21^{**}	2.48^{**}	1.24^{**}	$\gamma > 0$
UK	1957	2002	-0.16	-1.03^{**}	8.58^{**}	3.29^{**}	1.53^{**}	$0 < \gamma < 241.0$
US	1949	2002	0.14	-0.03	14.60^{***}	5.84^{***}	2.92^{***}	$0 < \gamma < 207.4$

allocation advice. By contrast, higher c_{ls} makes it less likely that the correlation falls short of $\overline{H}$, the critical value needed to justify the investment horizon advice. However, inspection of Tables 1 and 2 shows that the individual correlations would have to be implausibly high for this condition not to be satisfied: even when $\gamma = 10$, the minimum correlation required is 0.6; for lower γ values, this rises above the maximum-possible value of 1.0.

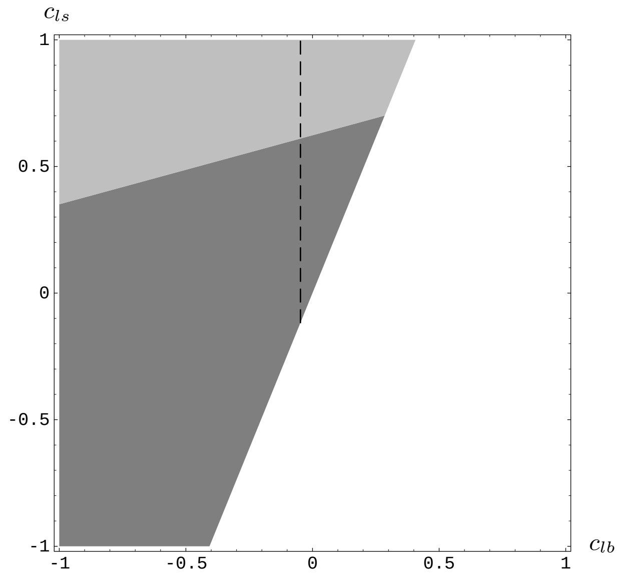
A further complication, although probably a second-order one, is that individual bond-labour correlations may also differ from the aggregate per-capita value. For some individuals, this correlation could be negative if their employment prospects are interest-rate sensitive. In any event, high c_{lb} makes it less likely that the asset allocation advice can be justified, but more likely that the investment horizon advice is optimal; low c_{lb} has the opposite effect.

Turning to the volatility effect, $\underline{H}$ is independent of σ_l, so higher income volatility on the part of individuals has no impact on the justification for the asset allocation advice. It does, however, change $\overline{H}$, albeit in an ambiguous manner that depends on the sign of $\mu_s - (c_{sb}\sigma_s/\sigma_b)\mu_b$. In our data, this term is positive, so higher income volatility makes it more difficult to justify the investment horizon advice.

To quantitatively assess the sensitivity of our results to individual income streams that are

Figure 1: Combinations of c_{lb} and c_{ls} that justify both pieces of investment advice

The two shaded areas together represent all combinations of c_{lb} and c_{ls} for which both the asset allocation advice and the investment horizon advice are justified (i.e., satisfy (10)) when σ_l is set equal to our estimate from aggregate 1930–1999 U.S. data. When σ_l is changed to three times its aggregate per-capita value, the allowed (c_{lb}, c_{ls}) combinations are now those falling in the dark shaded region only.



both systematically and unsystematically riskier than aggregate per capita income, we proceed in two steps. First, for σ_l equal to its aggregate per-capita value, we calculate the combinations of c_{lb} and c_{ls} which satisfy (10). Second, we increase σ_l by a factor of three and re-calculate the set of required (c_{lb}, c_{ls}) combinations.[16] We repeat this process for each country in our data set. Throughout, we set $\gamma = 5$.

The results for the 1930–1999 U.S. data (using long-term government bonds as the bond class; the other classes yield essentially the same results) are illustrated in Figure 1. The two shaded areas together represent all combinations of c_{ls} and c_{lb} for which both the asset allocation advice and the investment horizon advice are justified (i.e., satisfy (10)) when σ_l is set equal to our estimate from aggregate data. In this case, only the asset allocation constraint (7) turns out to be binding for these data; the investment horizon constraint (9) is satisfied for all possible values of c_{lb} and c_{ls}. As a result, even very high values of c_{ls} are feasible so long as c_{lb} is not

[16] The tripling of σ_l, although necessarily somewhat arbitrary, is based on Campbell et al. (2001). They estimate labour income variances for three different U.S. educational-achievement groups and report a maximum estimate approximately nine times our per-capita estimate for the U.S., so we increase our per-capita estimates for each country by the same factor.

also too positive. Overall, both pieces of investment advice are justified for any value of c_{ls} so long as $c_{lb} < -0.4$; every subsequent 10-point increase in c_{lb} rules out successively higher values of c_{ls} and raises the minimum c_{ls} by approximately 25 points. Once c_{lb} reaches about 0.4, the minimum c_{ls} exceeds 1 and the asset allocation advice cannot be justified.

Matters change somewhat when σ_l is changed to three times its aggregate per-capita value. Because this risk is non-tradable, other risky assets become less attractive and stocks must therefore be a better human-capital hedge (i.e., lower c_{ls}) in order to justify the investment horizon advice. In this case, the investment horizon constraint (9) also binds and the allowed (c_{lb}, c_{ls}) combinations are now those falling in the dark shaded region only. The principal effect of this is to eliminate some of the high c_{ls} values. Where previously, for example, c_{ls} as high as unity could justify both pieces of advice, now the maximum-possible value of c_{ls} ranges from 0.65 (when c_{lb} equals approximately 0.25) down to 0.35 (when c_{lb} equals -1).

One useful way of summarizing the information in plots like Figure 1 is given by the vertical dashed line. This depicts all values of c_{ls} that justify both pieces of investment advice when c_{lb} is equal to its aggregate per-capita value. For the 1930–1999 U.S. data, the “allowable” values for c_{ls} range from -0.12 to 1.00; this becomes -0.12 to 0.61 when σ_l is increased to three times its aggregate per-capita estimate. Both ranges seem likely to encompass the income processes for most individuals.[17]

For all other countries in our data set, we plot the corresponding vertical lines and report the resulting c_{ls} ranges in Table 3. For most countries, the allowable range of c_{ls} values is very wide indeed, and is reduced only slightly by significantly higher labour income volatility; only in Italy and Spain does the maximum-possible correlation fall below 0.5.

Finally, our model assumes that labour income is given by a simple exogenous process, thereby ruling out any flexibility in labour supply. However, as Bodie et al. (1992) point out, altering the supply of labour in response to economic conditions effectively allows investors to mitigate some of the risk associated with holding non-tradable human capital. This begs the question of how labour supply flexibility might alter our conclusions. Fortunately, this is straightforward to incorporate in our model, so we omit the details and focus on the intuition.[18]

The ability to adjust labour supply means that holdings of the human capital asset are no longer “fixed”, so there is less need to hedge the risk of this asset. This strengthens the case for the investment horizon advice: because labour supply can be varied, human capital is effectively less risky and thus younger investors do not need stocks to be such a good hedge in order to

[17] The maximum labour-stock correlation reported by Campbell et al. (2001) is 0.52.

[18] Section 6.1.2 of Campbell and Viceira (2002) outlines a simple method for incorporating flexible labour supply in portfolio models, and our discussion is based on that procedure.

Table 3: Range of c_{ls} values that justify both pieces of investment advice

This table reports the range of c_{ls} values that justify both pieces of investment advice when c_{lb} is equal to its aggregate per-capita value and $\gamma = 5$. Data are the same as in Table 2.

	σ_l = aggregate per-capita estimate	σ_l = 3×aggregate per-capita estimate
Australia	$-0.68 \leq c_{ls} \leq 1.00$	$-0.68 \leq c_{ls} \leq 1.00$
Canada	$0.09 \leq c_{ls} \leq 1.00$	$0.09 \leq c_{ls} \leq 1.00$
France	$-0.07 \leq c_{ls} \leq 1.00$	$-0.07 \leq c_{ls} \leq 0.62$
Ireland	$0.08 \leq c_{ls} \leq 1.00$	$0.08 \leq c_{ls} \leq 1.00$
Italy	$-0.21 \leq c_{ls} \leq 1.00$	$-0.21 \leq c_{ls} \leq 0.38$
Japan	$-0.18 \leq c_{ls} \leq 1.00$	$-0.18 \leq c_{ls} \leq 0.59$
Netherlands	$-0.12 \leq c_{ls} \leq 1.00$	$-0.12 \leq c_{ls} \leq 1.00$
Spain	$-0.52 \leq c_{ls} \leq 1.00$	$-0.52 \leq c_{ls} \leq 0.39$
Sweden	$-0.08 \leq c_{ls} \leq 1.00$	$-0.08 \leq c_{ls} \leq 0.82$
UK	$-0.94 \leq c_{ls} \leq 1.00$	$-0.94 \leq c_{ls} \leq 1.00$
US	$-0.03 \leq c_{ls} \leq 1.00$	$-0.03 \leq c_{ls} \leq 1.00$

justify holding more of them. Expressed in terms of our model parameters, $\overline{H}$ rises. By contrast, the asset allocation advice is unaffected: the relative demand for risky assets depends on the relative strengths of these assets in hedging labour income, a feature that is unaltered by labour supply flexibility.[19]

V Further issues

Our model can also be used to address other issues related to the two pieces of investment advice. For example, the investment horizon advice explicitly states that the allocation to stocks should decrease as the investment horizon shortens, but is less forthcoming on how this should affect the allocation among other assets. Is "stocks" implicit shorthand for all risky assets, so that the advice implies a reduction in both stocks and bonds, and therefore an increase in riskless cash, over a shorter time horizon? Or does "stocks" really mean stocks, leaving open the possibility that older investors should actually hold more bonds, even as they hold fewer stocks?

To address this issue, we calculate $\partial \alpha_b / \partial z$. From (4), this has the sign of $\overline{J} - c_{lb}$, where

$$\overline{J} = (1/\gamma\sigma_l\sigma_b)(\mu_b - (c_{sb}\sigma_b/\sigma_s)\mu_s) + c_{sb}c_{ls}.$$

If $c_{lb} < \overline{J}$, then bonds are a good hedge for non-tradable risky labour income and so fewer are

[19]Of course, the crucial correlations are now with real wage shocks rather than labour income; the important point is that condition (7) is otherwise unchanged.

Table 4: International estimates of c_{lb} and $\overline{J}$

Young investors hold relatively more of their wealth in bonds than old investors if and only if $c_{lb} < \overline{J}$, where $\overline{J}$ is a constant that depends on market and investor-specific parameters. Data are the same as in Table 2.

		$\overline{J}$		
	c_{lb}	$\gamma = 2$	$\gamma = 5$	$\gamma = 10$
Australia	-0.18	0.22	0.05	-0.01
Canada	0.03	2.66	1.06	0.53
France	-0.10	10.36	4.17	2.11
Ireland	0.03	-1.14	-0.35	-0.08
Italy	-0.30	5.33	2.11	1.04
Japan	-0.11	2.62	1.05	0.53
Netherlands	-0.03	0.94	0.37	0.18
Spain	-0.23	-0.81	-0.37	-0.22
Sweden	-0.04	2.03	0.81	0.40
UK	-0.39	-0.56	-0.25	-0.15
USA	-0.01	3.83	1.54	0.78

needed as the investment horizon shortens. But if $c_{lb} > \overline{J}$, then bonds are a good substitute for human capital and older investors optimally hold more of them.

Using our multi-country data, we calculate $\overline{J}$ for each country and compare this with the corresponding estimate of c_{lb}. The results appear in Table 4. For $\gamma = 5$, $\overline{J}$ comfortably exceeds c_{lb} for all countries except Spain, suggesting that the investment horizon advice usually applied to stocks also applies to bonds. Human capital is closer to an investment in cash than it is to an investment in stocks or bonds.

Returning to the optimal allocation of wealth between different risky assets, the asset allocation advice states that a lowering of investor tolerance towards a given level of risk should result in substitution from stocks to bonds, but is silent on the optimal response to a change in risk for a given level of tolerance. Given the focus of this paper, we are particularly interested in the effect of a change in the risk of labour income. In this context, Campbell and Viceria (2002) show that higher labour income risk lowers the allocation to risky assets if investors are sufficiently risk averse. Our model allows us to take this a step further and determine the effect of higher labour income risk (σ_l) on the allocation among risky assets.

The first point to note is that if labour income is idiosyncratic, then

$$\alpha = \frac{\sigma_b^2 \mu_s - \sigma_{sb}\mu_b}{\sigma_s^2 \mu_b - \sigma_{sb}\mu_s},$$

which is independent of σ_l. An increase in the risk of the non-traded component of the portfolio induces the investor to reduce the allocation to risky assets, but because the risk is idiosyncratic, the two risky assets are equivalent hedges and hence their relative allocation is unaffected. The risk of an investor's human capital affects the allocation between riskless and risky assets, but not the composition of the risky asset portfolio itself. Thus, we have an alternative separation result: among investors with identical risk tolerance but differing levels of idiosyncratic human capital risk, the portfolio risk decision can be separated from the risky asset allocation decision.

If labour income is not idiosyncratic, so that a change in σ_l also changes the covariances between asset returns and labour income, then matters become more complicated. In the appendix, we show that $d\alpha/d\sigma_l$ has the same sign as

$$\theta(\mu_s\sigma_{lb} - \mu_b\sigma_{ls}), \tag{11}$$

where

$$\theta = 1 + (1-\rho)\phi_p - \rho\sigma_l^2.$$

We recognise the right-hand term in (11) as being identical to (6), so this determines which risky asset is the closest substitute for human capital (i.e., has the highest relative correlation with labour income). Thus, if θ is positive, then an increase in non-tradable labour income risk reduces demand for the tradable risky asset that is most like human capital. However, the sign of θ is ambiguous, reflecting two effects that shift demand in opposite directions. First, faced with an increase in labour income risk, there is a standard substitution effect: risk-averse investors rebalance their portfolios away from the human capital substitute towards the human capital hedge. Second, there is an implicit income effect: with expected labour income held constant, the rise in σ_l lowers the mean of log labour income. This lowers the human capital allocation in the portfolio, thereby leading to *greater* demand for the tradable risky asset that is the closest substitute for human capital. Overall, because the magnitude of the first effect is proportional to $1 + (1-\rho)\phi_p$ while the second is proportional to $\rho\sigma_l^2$, an increase in σ_l leads to a lower allocation for the risky asset that is the closest substitute for human capital if and only if $\rho\sigma_l^2 < 1 + (1-\rho)\phi_p$.[20]

In our data, this condition is comfortably satisfied for all countries, while Tables 1 and 2 indicate that $\mu_s\sigma_{lb} - \mu_b\sigma_{ls}$ is negative. Thus, greater labour income risk should result in a reallocation of risk capital from stocks to bonds.

[20] If $\phi_p < -1/(1-\rho)$, then the substitution effect also increases demand for the tradable risky asset that is most like human capital, for reasons similar to those discussed in footnote 7.

VI Concluding remarks

Can human capital considerations resolve the asset allocation puzzle of Canner et al. (1997)? Those authors are doubtful, primarily because the strong correlation between stock returns and labour income gains that would be required also implies that investors with a long investment horizon should allocate less of their financial wealth to stocks, exactly the opposite of popular investment advice. However, once non-tradable human capital is explicitly modelled, the optimal stock-bond ratio depends not only on the correlations of these assets with labour income, but also on the simple risk-return trade-offs offered by these assets. As a result, the correlation between stock returns and labour income gains required to resolve the asset allocation puzzle does not, after all, have to be all that high, leaving open the possibility that it can be sufficiently low to also justify the investment horizon advice.

The principal contributions of this paper have been, first, to confirm the theoretical validity of the above logic, and second, to assess its empirical validity using historical data from a number of countries. The results of the latter exercise are somewhat ambiguous. Although the critical inequalities identified by our model are almost always evident in data based on aggregate per-capita labour income, the imprecision of our parameter estimates means that a number of these inequalities are statistically insignificant. In particular, the correlation between stock returns and labour income is a little too close to its lower bound, thereby making it difficult to conclude that this bound is truly satisfied in most cases. However, individual income processes are likely to differ significantly from the aggregate per-capita process, and a wide range of plausible processes satisfy the critical inequalities.

By including two risky assets, our model also permits more detailed analysis of optimal wealth allocation. We have focused on the effect of investment horizon on the demand for bonds and on the sensitivity of the stock/bond ratio to the volatility of labour income, but consideration of other issues may be a useful line of future research.

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Appendix

Proof of (3) and (4)

Maximizing the expected value of (1) subject to (2) yields the first-order conditions

$$E[(R_i - R_f)W^{-\gamma}] = 0, \quad i = s,\, b,$$

which can be re-written as

$$\log E[(1 + R_i)W^{-\gamma}] = r_f + \log E[W^{-\gamma}], \quad i = s,\, b, \tag{A-1}$$

where $r_f = \log(1 + R_f)$. To make this problem analytically tractable, we use two loglinear approximations developed by Campbell and Viceira (2001, 2002). First, a Taylor expansion of the logarithmic form of (2) gives

$$w \approx k + \rho(\overline{w} + r_p) + (1 - \rho)l, \tag{A-2}$$

where $r_p = \log(1 + R_p)$, $l = \log L$, $\overline{w} = \log \overline{W}$,

$$\rho = \frac{\exp(\overline{w} + E[r_p - l])}{1 + \exp(\overline{w} + E[r_p - l])} < 1,$$

and

$$k = \log(1 + \exp(\overline{w} + E[r_p - l])) - \rho(\overline{w} + E[r_p - l]).$$

Second, a Taylor expansion of $\log(1 + R_p)$ yields

$$r_p \approx r_f + \alpha_s(r_s - r_f) + \alpha_b(r_b - r_f) + \frac{1}{2}\left(\alpha_s(1 - \alpha_s)\sigma_s^2 - 2\alpha_s\alpha_b\sigma_{sb} + \alpha_b(1 - \alpha_b)\sigma_b^2\right), \tag{A-3}$$

where $r_i = \log(1 + R_i)$, $\sigma_i^2 = \mathrm{Var}[r_i]$, and $\sigma_{sb} = \mathrm{Cov}[r_s, r_b]$.

As r_s and r_b are jointly normal, (A-3) implies that r_p also has a normal distribution. Then, since l is also normal, (A-2) implies that w is normal as well. Thus, both terms inside the expectations operator in (A-1) are lognormally distributed. Using the standard properties of a lognormal random variable, we obtain

$$\begin{aligned}
\log E[(1+R_i)W^{-\gamma}] &= E[r_i - \gamma w] + \frac{1}{2}\mathrm{Var}[r_i - \gamma w], \quad i = s,\, b, \\
r_f + \log E[W^{-\gamma}] &= r_f - \gamma E[w] + \frac{1}{2}\mathrm{Var}[-\gamma w].
\end{aligned}$$

Substituting these back into (A-1) yields

$$\begin{aligned}
E[r_i] - r_f + \frac{\sigma_i^2}{2} &= \gamma\,\mathrm{Cov}[r_i, w] \\
&= \gamma\,\mathrm{Cov}[r_i, \rho r_p + (1-\rho)l] \\
&= \gamma\,\rho\mathrm{Cov}[r_i, \alpha_s r_s + \alpha_b r_b] + \gamma(1-\rho)\mathrm{Cov}[r_i, l],
\end{aligned}$$

where we used (A-2) and (A-3). This is a system of two linear equations in the two unknowns α_s and α_b. Solving this system produces (3) and (4).

Proof of (5)

From the first order conditions,

$$\alpha_i\mu_i = \gamma\rho\alpha_i^2\sigma_i^2 + \gamma\rho\alpha_s\alpha_b\sigma_{sb} + \gamma(1-\rho)\alpha_i\sigma_{li}.$$

Therefore

$$\begin{aligned}
\phi_p &= \alpha_s\frac{\partial E[r_p]}{\partial\alpha_s} + \alpha_b\frac{\partial E[r_p]}{\partial\alpha_b} \\
&= \alpha_s(\mu_s - \alpha_s\sigma_s^2 - \alpha_b\sigma_{sb}) + \alpha_b(\mu_b - \alpha_s\sigma_{sb} - \alpha_b\sigma_b^2) \\
&= \alpha_s\mu_s + \alpha_b\mu_b - (\alpha_s^2\sigma_s^2 + 2\alpha_s\alpha_b\sigma_{sb} + \alpha_b^2\sigma_b^2) \\
&= \gamma\rho\alpha_s^2\sigma_s^2 + \gamma\rho\alpha_s\alpha_b\sigma_{sb} + \gamma(1-\rho)\alpha_s\sigma_{ls} + \gamma\rho\alpha_b^2\sigma_b^2 + \gamma\rho\alpha_s\alpha_b\sigma_{sb} + \gamma(1-\rho)\alpha_b\sigma_{lb} \\
&\quad - (\alpha_s^2\sigma_s^2 + 2\alpha_s\alpha_b\sigma_{sb} + \alpha_b^2\sigma_b^2) \\
&= (\gamma\rho - 1)(\alpha_s^2\sigma_s^2 + 2\alpha_s\alpha_b\sigma_{sb} + \alpha_b^2\sigma_b^2) + \gamma(1-\rho)(\alpha_s\sigma_{ls} + \alpha_b\sigma_{lb}) \\
&= (\gamma\rho - 1)\mathrm{Var}[r_p] + \gamma(1-\rho)\mathrm{Cov}[r_p, l] \\
&= \mathrm{Var}[r_p](\gamma(\rho + (1-\rho)\beta) - 1),
\end{aligned}$$

where $\beta = \mathrm{Cov}[r_p, l]/\mathrm{Var}[r_p]$.

Proof of (6)

The following lemma will be useful in proving (6), (8), and (11):

Lemma 1 *The second order conditions of expected utility maximization imply that*

$$1 - \frac{\partial \alpha_s}{\partial \rho}\frac{\partial \rho}{\partial \alpha_s} - \frac{\partial \alpha_b}{\partial \rho}\frac{\partial \rho}{\partial \alpha_b} > 0.$$

Proof. Recall that the investor's objective function is

$$F(\alpha_s, \alpha_b) = E\left[\frac{W^{1-\gamma}}{1-\gamma}\right],$$

implying the first order conditions

$$\frac{\partial F}{\partial \alpha_i} = E[W^{-\gamma}(R_i - R_f)] = 0, \quad i = s, b.$$

Using the lognormality of returns and labour income,

$$\begin{aligned}
\frac{\partial F}{\partial \alpha_i} \approx G_i &\equiv E[\exp(r_i - \gamma w)] - E[\exp(r_f - \gamma w)] \\
&= \exp\left(E[r_i - \gamma w] + \frac{1}{2}\mathrm{Var}[r_i - \gamma w]\right) - \exp\left(E[r_f - \gamma w] + \frac{1}{2}\mathrm{Var}[-\gamma w]\right) \\
&= \exp\left(\mu_i + r_f - \gamma E[w] - \gamma \mathrm{Cov}[r_i, w] + \frac{\gamma^2}{2}\mathrm{Var}[w]\right) - \exp\left(r_f - \gamma E[w] + \frac{\gamma^2}{2}\mathrm{Var}[w]\right) \\
&= \exp\left(r_f - \gamma E[w] + \frac{\gamma^2}{2}\mathrm{Var}[w]\right)\left(\exp\left(\mu_i - \gamma \mathrm{Cov}[r_i, w]\right) - 1\right) \\
&= \exp(A)(\exp(B_i) - 1),
\end{aligned}$$

where

$$A = r_f - \gamma E[w] + \frac{\gamma^2}{2}\mathrm{Var}[w]$$

and

$$B_i = \mu_i - \gamma \mathrm{Cov}[r_i, w] = \mu_i - \gamma\rho\alpha_s\sigma_{is} - \gamma\rho\alpha_b\sigma_{ib} - \gamma(1-\rho)\sigma_{li}.$$

Since the first order conditions reduce to $B_i = 0$, we have

$$\frac{\partial G_i}{\partial \alpha_j} = \exp(A)\frac{\partial B_i}{\partial \alpha_j} \quad \text{and} \quad \frac{\partial G_i}{\partial \rho} = \exp(A)\frac{\partial B_i}{\partial \rho}$$

when evaluated at the optimal portfolio. Furthermore, since ρ (via $E[r_p]$) is a function of (α_s, α_b), the second order derivatives of F satisfy

$$\frac{\partial^2 F}{\partial \alpha_j \partial \alpha_i} \approx \frac{\partial G_i}{\partial \alpha_j} + \frac{\partial G_i}{\partial \rho}\frac{\partial \rho}{\partial \alpha_j} = \exp(A)\left(\frac{\partial B_i}{\partial \alpha_j} + \frac{\partial B_i}{\partial \rho}\frac{\partial \rho}{\partial \alpha_j}\right).$$

The second order conditions for expected utility maximization therefore imply that

$$\left(\frac{\partial B_s}{\partial \alpha_s} + \frac{\partial B_s}{\partial \rho}\frac{\partial \rho}{\partial \alpha_s}\right)\left(\frac{\partial B_b}{\partial \alpha_b} + \frac{\partial B_b}{\partial \rho}\frac{\partial \rho}{\partial \alpha_b}\right) - \left(\frac{\partial B_s}{\partial \alpha_b} + \frac{\partial B_s}{\partial \rho}\frac{\partial \rho}{\partial \alpha_b}\right)\left(\frac{\partial B_b}{\partial \alpha_s} + \frac{\partial B_b}{\partial \rho}\frac{\partial \rho}{\partial \alpha_s}\right) > 0.$$

If we expand both terms and rearrange, we obtain the following equivalent condition:

$$\frac{\partial B_s}{\partial \alpha_s}\frac{\partial B_b}{\partial \alpha_b} - \frac{\partial B_s}{\partial \alpha_b}\frac{\partial B_b}{\partial \alpha_s} + \frac{\partial \rho}{\partial \alpha_s}\left(\frac{\partial B_b}{\partial \alpha_b}\frac{\partial B_s}{\partial \rho} - \frac{\partial B_s}{\partial \alpha_b}\frac{\partial B_b}{\partial \rho}\right) + \frac{\partial \rho}{\partial \alpha_b}\left(\frac{\partial B_s}{\partial \alpha_s}\frac{\partial B_b}{\partial \rho} - \frac{\partial B_b}{\partial \alpha_s}\frac{\partial B_s}{\partial \rho}\right) > 0. \quad \text{(A-4)}$$

Now return to the first order conditions $B_i = 0$ for $i = s, b$. In the main text we solve these two equations to obtain (3) and (4), which give α_s and α_b as functions of ρ (and the various model parameters). That is, we effectively treat the first order conditions as equations of the form

$$0 = B_i(\alpha_s(\rho), \alpha_b(\rho), \rho), \quad i = s, b.$$

Differentiating with respect to ρ implies that

$$0 = \frac{\partial B_s}{\partial \rho} + \frac{\partial B_s}{\partial \alpha_s}\frac{\partial \alpha_s}{\partial \rho} + \frac{\partial B_s}{\partial \alpha_b}\frac{\partial \alpha_b}{\partial \rho}$$

and

$$0 = \frac{\partial B_b}{\partial \rho} + \frac{\partial B_b}{\partial \alpha_s}\frac{\partial \alpha_s}{\partial \rho} + \frac{\partial B_b}{\partial \alpha_b}\frac{\partial \alpha_b}{\partial \rho}.$$

Thus

$$\frac{\partial B_b}{\partial \alpha_b}\frac{\partial B_s}{\partial \rho} - \frac{\partial B_s}{\partial \alpha_b}\frac{\partial B_b}{\partial \rho} = -\left(\frac{\partial B_s}{\partial \alpha_s}\frac{\partial B_b}{\partial \alpha_b} - \frac{\partial B_s}{\partial \alpha_b}\frac{\partial B_b}{\partial \alpha_s}\right)\frac{\partial \alpha_s}{\partial \rho}$$

and

$$\frac{\partial B_s}{\partial \alpha_s}\frac{\partial B_b}{\partial \rho} - \frac{\partial B_b}{\partial \alpha_s}\frac{\partial B_s}{\partial \rho} = -\left(\frac{\partial B_s}{\partial \alpha_s}\frac{\partial B_b}{\partial \alpha_b} - \frac{\partial B_s}{\partial \alpha_b}\frac{\partial B_b}{\partial \alpha_s}\right)\frac{\partial \alpha_b}{\partial \rho}.$$

Condition (A-4) then implies that

$$\left(\frac{\partial B_s}{\partial \alpha_s}\frac{\partial B_b}{\partial \alpha_b} - \frac{\partial B_s}{\partial \alpha_b}\frac{\partial B_b}{\partial \alpha_s}\right)\left(1 - \frac{\partial \rho}{\partial \alpha_s}\frac{\partial \alpha_s}{\partial \rho} - \frac{\partial \rho}{\partial \alpha_b}\frac{\partial \alpha_b}{\partial \rho}\right) > 0.$$

Finally, note that

$$\frac{\partial B_i}{\partial \alpha_j} = -\gamma\rho\sigma_{ij},$$

which implies that

$$\frac{\partial B_s}{\partial \alpha_s}\frac{\partial B_b}{\partial \alpha_b} - \frac{\partial B_s}{\partial \alpha_b}\frac{\partial B_b}{\partial \alpha_s} = \gamma^2\rho^2(\sigma_s^2\sigma_b^2 - \sigma_{sb}^2) > 0.$$

Therefore, we have

$$1 - \frac{\partial \alpha_s}{\partial \rho}\frac{\partial \rho}{\partial \alpha_s} - \frac{\partial \alpha_b}{\partial \rho}\frac{\partial \rho}{\partial \alpha_b} > 0.$$

This completes the proof of Lemma 1. ■

Differentiating the equations $\alpha_i = \alpha_i(\gamma, \rho(\alpha_s, \alpha_b))$, $i = s, b$, with respect to γ shows that

$$\frac{d\alpha_i}{d\gamma} = \frac{\partial \alpha_i}{\partial \gamma} + \frac{\partial \alpha_i}{\partial \rho}\left(\frac{\partial \rho}{\partial \alpha_s}\frac{d\alpha_s}{d\gamma} + \frac{\partial \rho}{\partial \alpha_b}\frac{d\alpha_b}{d\gamma}\right), \quad i = s, b.$$

Solving this linear system of two equations for $\frac{d\alpha_s}{d\gamma}$ and $\frac{d\alpha_b}{d\gamma}$, we get

$$\frac{d\alpha_s}{d\gamma} = \frac{\frac{\partial \alpha_s}{\partial \gamma} + \frac{\partial \rho}{\partial \alpha_b}\left(\frac{\partial \alpha_s}{\partial \rho}\frac{\partial \alpha_b}{\partial \gamma} - \frac{\partial \alpha_b}{\partial \rho}\frac{\partial \alpha_s}{\partial \gamma}\right)}{1 - \frac{\partial \alpha_s}{\partial \rho}\frac{\partial \rho}{\partial \alpha_s} - \frac{\partial \alpha_b}{\partial \rho}\frac{\partial \rho}{\partial \alpha_b}}$$

and

$$\frac{d\alpha_b}{d\gamma} = \frac{\frac{\partial \alpha_b}{\partial \gamma} + \frac{\partial \rho}{\partial \alpha_s}\left(\frac{\partial \alpha_b}{\partial \rho}\frac{\partial \alpha_s}{\partial \gamma} - \frac{\partial \alpha_s}{\partial \rho}\frac{\partial \alpha_b}{\partial \gamma}\right)}{1 - \frac{\partial \alpha_s}{\partial \rho}\frac{\partial \rho}{\partial \alpha_s} - \frac{\partial \alpha_b}{\partial \rho}\frac{\partial \rho}{\partial \alpha_b}}.$$

Defining $\alpha = \alpha_s/\alpha_b$, it follows that

$$\begin{aligned}
\frac{d\alpha}{d\gamma} &= \frac{1}{(\alpha_b)^2}\left(\alpha_b\frac{d\alpha_s}{d\gamma} - \alpha_s\frac{d\alpha_b}{d\gamma}\right) \\
&= \frac{1}{(\alpha_b)^2}\left(\frac{\alpha_b\frac{\partial\alpha_s}{\partial\gamma} - \alpha_s\frac{\partial\alpha_b}{\partial\gamma} + \left(\frac{\partial\alpha_s}{\partial\rho}\frac{\partial\alpha_b}{\partial\gamma} - \frac{\partial\alpha_s}{\partial\gamma}\frac{\partial\alpha_b}{\partial\rho}\right)\left(\alpha_b\frac{\partial\rho}{\partial\alpha_b} + \alpha_s\frac{\partial\rho}{\partial\alpha_s}\right)}{1 - \frac{\partial\alpha_s}{\partial\rho}\frac{\partial\rho}{\partial\alpha_s} - \frac{\partial\alpha_b}{\partial\rho}\frac{\partial\rho}{\partial\alpha_b}}\right).
\end{aligned}$$

From Lemma 1, the denominator is positive, so that $\frac{d\alpha}{d\gamma}$ has the same sign as the numerator. Now,

$$\alpha_b\frac{\partial\alpha_s}{\partial\gamma} - \alpha_s\frac{\partial\alpha_b}{\partial\gamma} = \frac{1-\rho}{\gamma^2\rho^2\Delta}(\mu_s\sigma_{lb} - \mu_b\sigma_{ls}),$$

and

$$\frac{\partial\alpha_s}{\partial\rho}\frac{\partial\alpha_b}{\partial\gamma} - \frac{\partial\alpha_s}{\partial\gamma}\frac{\partial\alpha_b}{\partial\rho} = \frac{\mu_s\sigma_{lb} - \mu_b\sigma_{ls}}{\gamma^2\rho^3\Delta}$$

where we have used (3) and (4) to calculate the partial derivatives. Further, because $\frac{\partial\rho}{\partial E[r_p]} = \rho(1-\rho)$,

$$\frac{\partial\rho}{\partial\alpha_i} = \rho(1-\rho)\frac{\partial E[r_p]}{\partial\alpha_i}.$$

Therefore, the numerator of the expression for $\frac{d\alpha}{d\gamma}$ equals

$$\frac{(1-\rho)(1+\phi_p)}{\gamma^2\rho^2\Delta}(\mu_s\sigma_{lb} - \mu_b\sigma_{ls}).$$

Since $\rho < 1$ and $\Delta > 0$, $\frac{d\alpha}{d\gamma}$ has the same sign as $(1+\phi_p)(\mu_s\sigma_{lb} - \mu_b\sigma_{ls})$.

Proof of (8)

Differentiating the equations $\alpha_i = \alpha_i(\rho(\alpha_s, \alpha_b, z))$, $i = s, b$, with respect to z shows that

$$\frac{d\alpha_i}{dz} = \frac{\partial\alpha_i}{\partial\rho}\left(\frac{\partial\rho}{\partial z} + \frac{\partial\rho}{\partial\alpha_s}\frac{d\alpha_s}{dz} + \frac{\partial\rho}{\partial\alpha_b}\frac{d\alpha_b}{dz}\right), \quad i = s, b.$$

Solving this linear system of two equations for $\frac{d\alpha_s}{dz}$ and $\frac{d\alpha_b}{dz}$, we get

$$\frac{d\alpha_s}{dz} = \frac{\frac{\partial\alpha_s}{\partial\rho}\frac{\partial\rho}{\partial z}}{1 - \frac{\partial\alpha_s}{\partial\rho}\frac{\partial\rho}{\partial\alpha_s} - \frac{\partial\alpha_b}{\partial\rho}\frac{\partial\rho}{\partial\alpha_b}}$$

and

$$\frac{d\alpha_b}{dz} = \frac{\frac{\partial\alpha_b}{\partial\rho}\frac{\partial\rho}{\partial z}}{1 - \frac{\partial\alpha_s}{\partial\rho}\frac{\partial\rho}{\partial\alpha_s} - \frac{\partial\alpha_b}{\partial\rho}\frac{\partial\rho}{\partial\alpha_b}}.$$

From Lemma 1, the denominator of these two expressions is positive, so that $\frac{d\alpha_s}{dz}$ has the same sign as

$$\frac{\partial\alpha_s}{\partial\rho}\frac{\partial\rho}{\partial z} = \frac{1-\rho}{\gamma\rho\Delta}\left(\sigma_b^2(\mu_s - \gamma\sigma_{ls}) - \sigma_{sb}(\mu_b - \gamma\sigma_{lb})\right),$$

where we have used (3) to calculate the first partial derivative. Since $0 < \rho < 1$ and $\Delta > 0$, $\frac{d\alpha_s}{dz}$ has the same sign as

$$\sigma_b^2(\mu_s - \gamma\sigma_{ls}) - \sigma_{sb}(\mu_b - \gamma\sigma_{lb}).$$

Similarly, $\frac{d\alpha_b}{dz}$ has the same sign as

$$\sigma_s^2(\mu_b - \gamma\sigma_{lb}) - \sigma_{sb}(\mu_s - \gamma\sigma_{ls}).$$

Proof of (11)

Differentiating the equations $\alpha_i = \alpha_i(\sigma_l, \rho(\alpha_s, \alpha_b, \sigma_l))$, $i = s, b$, with respect to σ_l shows that

$$\frac{d\alpha_i}{d\sigma_l} = \frac{\partial\alpha_i}{\partial\sigma_l} + \frac{\partial\alpha_i}{\partial\rho}\left(\frac{\partial\rho}{\partial\sigma_l} + \frac{\partial\rho}{\partial\alpha_s}\frac{d\alpha_s}{d\sigma_l} + \frac{\partial\rho}{\partial\alpha_b}\frac{d\alpha_b}{d\sigma_l}\right), \quad i = s, b.$$

Solving this linear system of two equations for $\frac{d\alpha_s}{d\sigma_l}$ and $\frac{d\alpha_b}{d\sigma_l}$, we get

$$\frac{d\alpha_s}{d\sigma_l} = \frac{\frac{\partial\alpha_s}{\partial\sigma_l} - \frac{\partial\alpha_b}{\partial\rho}\frac{\partial\alpha_s}{\partial\sigma_l}\frac{\partial\rho}{\partial\alpha_b} + \frac{\partial\alpha_b}{\partial\sigma_l}\frac{\partial\alpha_s}{\partial\rho}\frac{\partial\rho}{\partial\alpha_b} + \frac{\partial\alpha_s}{\partial\rho}\frac{\partial\rho}{\partial\sigma_l}}{1 - \frac{\partial\alpha_s}{\partial\rho}\frac{\partial\rho}{\partial\alpha_s} - \frac{\partial\alpha_b}{\partial\rho}\frac{\partial\rho}{\partial\alpha_b}}$$

and

$$\frac{d\alpha_b}{d\sigma_l} = \frac{\frac{\partial\alpha_b}{\partial\sigma_l} + \frac{\partial\alpha_b}{\partial\rho}\frac{\partial\alpha_s}{\partial\sigma_l}\frac{\partial\rho}{\partial\alpha_s} - \frac{\partial\alpha_b}{\partial\sigma_l}\frac{\partial\alpha_s}{\partial\rho}\frac{\partial\rho}{\partial\alpha_s} + \frac{\partial\alpha_b}{\partial\rho}\frac{\partial\rho}{\partial\sigma_l}}{1 - \frac{\partial\alpha_s}{\partial\rho}\frac{\partial\rho}{\partial\alpha_s} - \frac{\partial\alpha_b}{\partial\rho}\frac{\partial\rho}{\partial\alpha_b}}.$$

It follows that

$$\frac{d\alpha}{d\sigma_l} = \frac{\left(\alpha_b\frac{\partial\alpha_s}{\partial\sigma_l} - \frac{\partial\alpha_b}{\partial\sigma_l}\alpha_s\right) + \left(\alpha_b\frac{\partial\alpha_s}{\partial\rho} - \frac{\partial\alpha_b}{\partial\rho}\alpha_s\right)\frac{\partial\rho}{\partial\sigma_l} + \left(\frac{\partial\alpha_s}{\partial\rho}\frac{\partial\alpha_b}{\partial\sigma_l} - \frac{\partial\alpha_s}{\partial\sigma_l}\frac{\partial\alpha_b}{\partial\rho}\right)\left(\alpha_b\frac{\partial\rho}{\partial\alpha_b} + \alpha_s\frac{\partial\rho}{\partial\alpha_s}\right)}{\alpha_b^2(1 - \frac{\partial\alpha_s}{\partial\rho}\frac{\partial\rho}{\partial\alpha_s} - \frac{\partial\alpha_b}{\partial\rho}\frac{\partial\rho}{\partial\alpha_b})}.$$

Because $E[l] = \log E[L] - \sigma_l^2/2$,

$$\frac{\partial\rho}{\partial\sigma_l} = \sigma_l(1-\rho)\rho.$$

Further, because $\frac{\partial\rho}{\partial E[r_p]} = \rho(1-\rho)$,

$$\frac{\partial\rho}{\partial\alpha_i} = \rho(1-\rho)\frac{\partial E[r_p]}{\partial\alpha_i}.$$

After some algebraic manipulation, we find that the numerator of $\frac{d\alpha}{d\sigma_l}$ equals

$$\frac{(1-\rho)\theta}{\sigma_l\gamma\rho^2\Delta}\left(\sigma_{lb}\mu_s - \sigma_{ls}\,\mu_b\right),$$

where

$$\theta = 1 + (1-\rho)\phi_p - \rho\sigma_l^2.$$

Since $0 < \rho < 1$ and $\Delta > 0$, $\frac{d\alpha}{d\sigma_l}$ has the same sign as $\theta(\sigma_{lb}\mu_s - \sigma_{ls}\,\mu_b)$.